HEAT KERNEL EXPANSION FOR OPERATORS OF THE TYPE OF THE SQUARE ROOT OF THE LAPLACE OPERATOR

E.V. Gorbar

Bogolyubov Institute for Theoretical Physics
252143 Kiev, Ukraine

Abstract

A method is suggested for the calculation of the DeWitt-Seeley-Gilkey (DWSG) coefficients for the operator $\sqrt{-\nabla^2 + V(x)}$ basing on a generalization of the pseudodifferential operator technique. The lowest DWSG coefficients for the operator $\sqrt{-\nabla^2} + V(x)$ are calculated by using the method proposed. It is shown that the method admits a generalization to the case of operators of the type $(-\nabla^2 + V(X))^{1/m}$, where m is an arbitrary rational number. A more simple method is proposed for the calculation of the DWSG coefficients for the case of strictly positive operators under the sign of root. By using this method, it is shown that the problem of the calculation of the DWSG coefficients for such operators is exactly solvable. Namely, an explicit formula expressing the DWSG coefficients for operators with root through the DWSG coefficients for operators without root is deduced.

1 Introduction

The algorithms for obtaining the asymptotic heat kernel expansion for second order differential operators on a Riemannian manifold are well known [1-3]. The most popular is that of DeWitt [1,4] which uses a certain ansatz for heat kernel matrix elements. The method possesses the explicit covariance with respect to gauge and general-coordinate transformations. However, the DeWitt technique does not apply to higher-order operators, nonminimal operators and, generally speaking, to operators whose leading term is not a power of the Laplace operator. Recently, using the Widom generalization [5] of the pseudodifferential operator technique a new algorithm was developed [6] for computing the DeWitt-Seeley-Gilkey (DWSG) coefficients. The method is explicitly gauge and geometrically covariant and admits to carry out calculations of the DWSG coefficients by computer [7]. As was shown in [8,9], the method permits a generalization to the case of Riemann-Cartan manifolds, i.e., manifolds with torsion, and to the case of nonminimal differential operators. In this paper the method of ref. [6] is generalized to the case of operators of the type of the square root of the Laplace operator. In order that extraction of the root be meaningful, the operator under the sign of root should be nonnegative, i.e., eigenvalues should be only positive or zero. It is not an essential restriction as to the applicability of the method proposed because in physics we are mainly encountered with operators bounded from below. Up to our knoweledge, there is no available method in current literature for the calculation of the DWSG coefficients for such type of operators. In Section 2 we compute the lowest E_2 DWSG coefficient for the operator $\sqrt{-\nabla^2 + V(x)}$ by using a method proposed with the expansion of the root. In Section 3 we generalize the method to the case of an arbitrary natural root, i.e., for the operator $(\sqrt{-\nabla^2} + V(x))^{1/m}$ where m is any natural and also to case of the operator $\sqrt{-\nabla^2} + V(x)$ which cannot be represented as any power of the operator $-\nabla^2 + V(x)$. In Section 4, for the case of strictly positive operators under the sign of root, we propose a more simple method for the calculation of the DWSG coefficients. By using this method, we were able to shown that the problem of the calculation of the DWSG coefficients for strictly positive operators under the sign of root is exactly solvable. Namely, an explicit formula expressing the DWSG coeffcients for operators with root through those for operators without root is deduced.

2 Method for calculation of the DWSG coefficients for the operator of the square root of the Laplace operator

We take as our space a compact n-dimensional Riemann manifold M without a boundary. The operators will act on the space of a vector bundle over the base M. The covariant derivative acting on objects with fiber (left understood) and base indices is defined by the rule

$$\nabla_\mu \phi_{\mu_1...\mu_k} = (\partial_\mu + \omega_\mu)\phi_{\mu_1...\mu_k} - \sum_{i=1}^{k} \Gamma^\lambda_{\mu_i\mu}\phi_{\mu_1...\mu_{i-1}\lambda\mu_{i+1}...\mu_k} \qquad (1)$$

where $\Gamma^\lambda_{\mu\nu}$ and ω_μ are the affine and bundle connections; $\omega_\mu = -\frac{1}{2}i\omega_\mu^{ab}\Sigma_{ab} + iA_\mu$, and Σ_{ab} are the representation operators of local rotation group SO(n) under which $\phi_{\mu_1...\mu_k}$ is transformed, ω_μ^{ab} is a spin connection and A_μ is gauge potential. For the commutator of covariant derivatives we have

$$[\nabla_\mu, \nabla_\nu]\phi_{\mu_1...\mu_k} = -\sum_{i=1}^{k} R^\lambda_{\mu_i\mu\nu}\phi_{\mu_i...\mu_{i-1}\lambda\mu_{i+1}...\mu_k} + W_{\mu\nu}\phi_{\mu_1...\mu_k}, \qquad (2)$$

where $W_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu + [\omega_\mu, \omega_\nu]$ is the bundle curvature, and the Riemann curvature tensor $R^\lambda_{\rho\mu\nu}$ is expressed through the affine connection $\Gamma^\lambda_{\mu\nu}$ as follows:

$$R^\lambda_{\rho\mu\nu} = \partial_\mu\Gamma^\lambda_{\rho\nu} - \partial_\nu\Gamma^\lambda_{\rho\mu} + \Gamma^\lambda_{\sigma\mu}\Gamma^\sigma_{\rho\nu} - \Gamma^\lambda_{\sigma\nu}\Gamma^\sigma_{\rho\mu}. \qquad (3)$$

As it is well khown [2,3,10], for a positive elliptic differential operator A of the order $2r$ there exists an asymptotic expansion of the diagonal matrix elements of the heat kernel $\exp(-tA)$ as $t \to 0_+$ in the following form:

$$<x|e^{-tA}|x> \simeq \sum_m E_m(x|A)t^{(m-n)/2r}, \qquad (4)$$

where the summation is carried out over all non-negative integers m and $E_m(x|A)$ are the DWSG coefficients.

In this section we consider a generalization of the method of [6] to the case of the operator

$$A = \sqrt{-\nabla^2 + V(x)}, \qquad (5)$$

where V(x) is an arbitrary matrix with respect to bundle space indices. Following the method [6], to obtain expansion (4) we use the representation of the operator $\exp(-tA)$ $(t > 0)$ through the operator A resolvent

$$e^{-tA} = \int_C \frac{id\lambda}{2\pi} e^{-t\lambda} (A - \lambda)^{-1}, \tag{6}$$

where the countour C goes counterclockwise around the spectrum of the operator A. For the matrix elements of the resolvent $(A - \lambda)^{-1}$ we employ the representation in the form

$$< x|\frac{1}{A-\lambda}|x' >= \int \frac{d^n k}{(2\pi)^n} \frac{1}{\sqrt{g(x')}} e^{il(x,x',k)} \sigma(x, x', k; \lambda), \tag{7}$$

where $l(x, x', k)$ is a phase function and $\sigma(x, x', k; \lambda)$ is an amplitude [10, 11]. In the flat space the phase $l(x, x', k) = k_\mu (x - x')^\mu$ is a linear function of k and x for each x'. In the case of a curved manifold the real function l must be biscalar with respect to general-coordinate transformations, and must be a linear homogeneous function in k. The generalization of the linearity condition in x is the requirement for the mth symmetrized covariant derivative to vanish at the point x' with $m \geq 2$, i.e.

$$\{\nabla_{\mu_1} \nabla_{\mu_2} \ldots \nabla_{\mu_m}\} l|_{x=x'} = [\{\nabla_{\mu_1} \nabla_{\mu_2} \ldots \nabla_{\mu_m}\} l] =$$

$$k_{\mu_1} \text{ for } m = 1 \text{ and } 0 \text{ for } m \neq 1. \tag{8}$$

In eq. (8) the curly brackets denote symmetrization in all indices and the square brackets mean that the coincidence limit is taken. The local properties of the function l are sufficient to obtain the diagonal heat kernel expansion.

The resolvent of the operator A satisfies the equation

$$(A(x, \nabla_\mu) - \lambda) \Gamma(x, x', k; \lambda) = \frac{1}{\sqrt{g}} \delta(x - x'), \tag{9}$$

and, therefore, in order to fulfill (9) it is sufficient to require that the amplitude $\sigma(x, x', k; \lambda)$ satisfy the equation

$$(A(x, \nabla_\mu + i\nabla_\mu l) - \lambda) \sigma(x, x', k; \lambda) = I(x, x'). \tag{10}$$

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The biscalar function $I(x, x')$ is a matrix with respect to bundle space indices and is defined by the conditions similar to eq. (8):

$$[I] = 1,$$

$$[\{\nabla_{\mu_1}\nabla_{\mu_2}\ldots\nabla_{\mu_m}\} I] = 0 \ \ m \geq 1, \tag{11}$$

the unity in eq. (11) is a matrix unity. To generate expansion (4), we introduce an auxiliary parameter ϵ into eq. (10) according to the rule $l \to l/\epsilon$, $\lambda \to \lambda/\epsilon$, and expand the amplitude in a formal series in the powers of ϵ

$$\sigma(x, x', k; \lambda) = \sum_{m=0}^{\infty} \epsilon^{1+m} \sigma_m(x, x', k; \lambda) \tag{12}$$

(the parameter ϵ then set equal to one). Then, eq. (10) gives us the recursion equations to determine the coefficients σ_m, and, finally, this procedure leads to expansion (4) where the DWSG coefficients $E_m(x|A)$ are expressed through the integrals of $[\sigma_m]$ in the form [6]:

$$E_m(x|A) = \int \frac{d^n k}{(2\pi)^n \sqrt{g}} \int_C \frac{id\lambda}{2\pi} e^{-\lambda} [\sigma_m](x, x, k; \lambda). \tag{13}$$

Up to now we followed [6] very closely. Differences arise for the operator of the type of the square root of the Laplace operator when we are going to obtain the recursion relations for σ_m. For the ordinary Laplace operator the recursion relations for σ_m follow directly from eq. (10) but it is not a case for the operator $A = \sqrt{-\nabla^2 + V(x)}$ which we consider. Explicitly, the equation for σ takes the form

$$\left(\sqrt{\nabla_\mu l\nabla^\mu l - i\epsilon\nabla^2 l - \epsilon^2\nabla^2 - 2i\epsilon\nabla_\mu l\nabla^\mu + \epsilon^2 V(x)} - \lambda\right) \sum_{m=0}^{\infty} \epsilon^m \sigma_m = I \tag{14}$$

We cannot just expand the square root of the operator in the powers of ϵ in the Tailor series as in the case of the Laplace operator because $\nabla_\mu l\nabla^\mu l$ and the operator with ϵ and ϵ^2 do not commute and it is not clear in which order to place them in the Tailor formula. Therefore, to generate an expansion of the root in powers of ϵ we first write down a general structure for the expansion of the root in the powers of ϵ

$$\sqrt{\nabla_\mu l\nabla^\mu l - i\epsilon\nabla^2 l - \epsilon^2\nabla^2 - 2i\epsilon\nabla_\mu l\nabla^\mu + \epsilon^2 V(x)} = \sqrt{\nabla_\mu l\nabla^\mu l} + \epsilon f_1 +$$

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$$\epsilon^2 f_2 + \ldots + \epsilon^m f_m + \ldots \tag{15}$$

where f_1, f_2 and f_m are to be found. In order to show how the method proposed works, first we compute the lowest E_0 and E_2 DWSG coefficients. To do this, we have to find the expansion of the root up to ϵ^2, i.e. we have to find only f_1 and f_2. This can be done as follows: First, we take the square of eq. (15)

$$\nabla_\mu l \nabla^\mu l - i\epsilon \nabla^2 l - \epsilon^2 \nabla^2 - 2i\epsilon \nabla_\mu l \nabla^\mu + \epsilon^2 V(x) = \nabla_\mu l \nabla^\mu l +$$

$$\sqrt{\nabla_\mu l \nabla^\mu l}\,\epsilon f_1 + \epsilon f_1 \sqrt{\nabla_\mu l \nabla^\mu l} + \epsilon^2 f_1^2 + \sqrt{\nabla_\mu l \nabla^\mu l}\,\epsilon^2 f_2 + \epsilon^2 f_2 \sqrt{\nabla_\mu l \nabla^\mu l} + \ldots \tag{16}$$

Then, comparing terms with the equal powers of ϵ, we obtain the equations for f_1 and f_2

$$-i\nabla^2 l - 2i\nabla_\mu l \nabla^\mu = \sqrt{\nabla_\mu l \nabla^\mu l}\,f_1 + f_1 \sqrt{\nabla_\mu l \nabla^\mu l}, \tag{17}$$

$$-\nabla^2 + V(x) = f_1^2 + \sqrt{\nabla_\mu l \nabla^\mu l}\,f_2 + f_2 \sqrt{\nabla_\mu l \nabla^\mu l} \tag{18}$$

From the left-hand side of eq. (17) it follows that a general structure of f_1 is $f_1 = -ia_\mu \nabla^\mu - ib$, where a_μ and b are ordinary vector and scalar functions, respectively, not operators. Substituting the general expression for f_1 into eq. (17), we get the following equations for a_μ and b:

$$-2i\nabla_\mu l \nabla^\mu = -i\sqrt{\nabla_\nu l \nabla^\nu l}\,a_\mu \nabla^\mu - ia_\mu \sqrt{\nabla_\nu l \nabla^\nu l}\,\nabla^\mu \tag{19}$$

$$-i\nabla^2 l = -i\sqrt{\nabla_\mu l \nabla^\mu l}\,b - ib\sqrt{\nabla_\mu l \nabla^\mu l} - ia_\mu \nabla^\mu \sqrt{\nabla_\nu l \nabla^\nu l} \tag{20}$$

From these equations we obtain

$$a_\mu = \frac{\nabla_\mu l}{\sqrt{\nabla_\nu l \nabla^\nu l}},$$

$$b = \frac{\nabla^2 l}{2\sqrt{\nabla_\mu l \nabla^\mu l}} - \frac{\nabla_\mu l \nabla^\mu \sqrt{\nabla_\nu l \nabla^\nu l}}{2\nabla_\alpha l \nabla^\alpha l} \tag{21}$$

Finally, the equation for f_2 takes the form

$$-\nabla^2 + V(x) = -(a_\mu \nabla^\mu + b)^2 + \sqrt{\nabla_\mu l \nabla^\mu l}\,f_2 + f_2 \sqrt{\nabla_\mu l \nabla^\mu l} \tag{22}$$

Similarly to the case of f_1 we write down a general structure of f_2

$$f_2 = C_1 \nabla^2 + C_{2\mu} \nabla^\mu + C_{3\mu\nu} \nabla^\mu \nabla^\nu + C_4 \qquad (23)$$

Substituting (22) into eq.(24), we find

$$C_1 = -\frac{1}{2R^{1/2}},$$

$$C_{2\mu} = \frac{\nabla_\mu R^{1/2}}{2R} - \frac{a_\mu a_\nu \nabla^\nu R^{1/2}}{2R} + \frac{a_\mu b}{R^{1/2}} + \frac{a_\nu \nabla^\nu a_\mu}{2R^{1/2}},$$

$$C_{3\mu\nu} = \frac{a_\mu a_\nu}{2R^{1/2}},$$

$$C_4 = \frac{V(x)}{2R^{1/2}} + \frac{a_\mu \nabla^\mu b}{R^{1/2}} + \frac{b^2}{2R^{1/2}} - \frac{\nabla^2 R^{1/2}}{4R} - \frac{1}{2R^{1/2}}\left(\frac{\nabla_\mu R^{1/2} \nabla^\mu R^{1/2}}{2R} - \right.$$

$$\left. \frac{a_\nu a_\mu \nabla^\nu R^{1/2} \nabla^\mu R^{1/2}}{2R} + \frac{a_\mu b \nabla^\mu R^{1/2}}{R^{1/2}} + \frac{a_\nu \nabla^\nu a_\mu \nabla^\mu R^{1/2}}{2R^{1/2}}\right) - \frac{a_\mu a_\nu \nabla^\mu \nabla^\nu R^{1/2}}{2R}, \quad (24)$$

where $R = \nabla_\mu l \nabla^\mu l$.

Thus, we have found the expansion of the root up to ϵ^2 but it is obviously that it is possible in a similar way to find the expansion of the root up to any mth power of ϵ because the equation for f_m has a similar form to the equations for f_1 and f_2, namely, $\sqrt{\nabla_\mu l \nabla^\mu l} f_m + f_m \sqrt{\nabla_\mu l \nabla^\mu l}$. Consequently, writting down a general stucture of f_m and defining f_1, f_2,..., f_{m-1}, we can find the explicit expression for f_m in the same way as it was done in the case of f_1 and f_2.

Having obtained the explicit expansion of the root

$$\sqrt{\nabla_\mu l \nabla^\mu l - i\epsilon \nabla^2 l - \epsilon^2 \nabla^2 - 2i\epsilon \nabla_\mu l \nabla^\mu + \epsilon^2 V(x)} =$$

$$\sqrt{\nabla_\mu l \nabla^\mu l} - i\epsilon(a_\mu \nabla^\mu + b) + \epsilon^2 f_2 + \dots, \qquad (25)$$

from eq.(14) we have the following equations for σ_0, σ_1 and σ_2:

$$(R^{1/2} - \lambda)\sigma_0 = I,$$

$$(R^{1/2} - \lambda)\sigma_1 - i(a_\mu \nabla^\mu + b)\sigma_0 = 0,$$

$$(R^{1/2} - \lambda)\sigma_2 - i(a_\mu \nabla^\mu + b)\sigma_1 + (C_1 \nabla^2 + C_{2\mu}\nabla^\mu + C_{3\mu\nu}\nabla^\mu\nabla^\nu + C_4)\sigma_0 = 0. \quad (26)$$

From eqn.(26) we obtain

$$[\sigma_0] = \frac{1}{\sqrt{k^2} - \lambda}, \quad (27)$$

$$[\sigma_2] = -\frac{k_\mu k^\lambda l^{\mu\nu}_{\ \ \nu\lambda}}{2k^2(\sqrt{k^2} - \lambda)^3} - \frac{k_\mu k_\lambda l^{\nu\ \mu\lambda}_{\ \nu}}{2k^2(\sqrt{k^2} - \lambda)^3} - \frac{V(x)}{2\sqrt{k^2}(\sqrt{k^2} - \lambda)^2} -$$

$$\frac{k_\mu k^\lambda l^{\mu\nu}_{\ \ nu\lambda}}{4(k^2)^{3/2}(\sqrt{k^2} - \lambda)^3} - \frac{k_\mu k_\lambda l^{\nu\ \mu\lambda}_{\ \nu}}{4(k^2)^{3/2}(\sqrt{k^2} - \lambda)^3}, \quad (28)$$

where we introduced the notation $[\nabla_\mu \nabla_\nu \ldots \nabla_\lambda l] = k_\alpha l_{\mu\nu\ldots\lambda\alpha}$ (see [6]) and wrote down only terms which do not vanish after the substitution of the explicit expression for $l_{\mu\nu\ldots\lambda}$ and the convolution with $k^\mu k^\nu \ldots k^\lambda$.

Let us recall that the DWSG coefficients are given by (13) and we have to calculate the integrals in λ and k. The integral in λ is trivially calculated by using the residue theory, and, consequently, we obtain

$$E_0(x) = \int \frac{d^n k}{(2\pi)^n \sqrt{g(x)}} e^{-\sqrt{k^2}}, \quad (29)$$

$$E_2(x) = \int \frac{d^n k}{(2\pi)^n \sqrt{g(x)}} e^{-\sqrt{k^2}} \left(-\frac{k_\mu k^\lambda l^{\mu\nu}_{\ \ \nu\lambda}}{4k^2} - \right.$$

$$\left. \frac{k_\mu k_\lambda l^{\nu\mu\lambda}_{\ \nu}}{4k^2} - \frac{V(x)}{2\sqrt{k^2}} - \frac{k_\mu k^\lambda l^{\mu\nu}_{\ \ \nu\lambda}}{4(k^2)^{3/2}} - \frac{k_\mu k_\lambda l^{\nu\mu\lambda}_{\ \nu}}{4(k^2)^{3/2}} \right). \quad (30)$$

To calculate the integral in k, we note that (see [6])

$$\int \frac{d^n k}{(2\pi)^n \sqrt{g}} k_{\mu_1} k_{\mu_2} \ldots k_{\mu_{2s}} f(k^2) =$$

$$g_{\{\mu_1\mu_2\ldots\mu_{2s}\}} \frac{1}{(4\pi)^{n/2} 2^s \Gamma(n/2 + s)} \int_0^\infty dk^2 (k^2)^{(n-2)/2+s} f(k^2),$$

$$k^2 = g^{\mu\nu} k_\mu k_\nu, \tag{31}$$

where $g_{\{\mu_1\mu_2...\mu_{2s}\}}$ is the symmetrized sum of metric tensor products. Integrating in k^2, we obtain

$$E_0(x) = \frac{2\Gamma(n)}{(4\pi)^{n/2}\Gamma(n/2)}, \tag{32}$$

$$E_2(x) = \frac{1}{(4\pi)^{n/2}} \Big(\frac{l_{\nu\mu}^{\mu\nu}\Gamma(n)}{4\Gamma(n/2+1)} - \frac{V(x)\Gamma(n-1)}{\Gamma(n/2)} - \frac{l_{\mu\nu}^{\mu\nu}\Gamma(n)}{4\Gamma(n/2+1)} -$$

$$\frac{l_{\nu\mu}^{\mu\nu}\Gamma(n-1)}{4\Gamma(n/2+1)} - \frac{l_{\mu\nu}^{\mu\nu}\Gamma(n-1)}{4\Gamma(n/2+1)} \Big). \tag{33}$$

Using $l_{\mu\nu\lambda\alpha} = -\frac{R_{\alpha\lambda\mu\nu}}{3} - \frac{R_{\alpha\nu\mu\lambda}}{3}$ [6], we obtain

$$E_2(x) = \frac{2\Gamma(n-2)}{(4\pi)^{n/2}\Gamma(n/2-1)} \left(\frac{R}{6} - V(x) \right). \tag{34}$$

Note that E_2 obtained for the operator of the type of the square root of the Laplace operator coincides with E_2 calculated for the Laplace operator up to a contant factor and, thus, the dependence of this coefficient on the space dimension is rather trivial (cf. with the case of nonmiminal operators [8] whose leading coefficients also are not a power of the Laplace operator). We will show in Section 4 that for strictly positive operators the same is true for the DWSG coefficients of an arbitrary order. It is an interesting problem whether it is also true for operators which have zero eigenmodes. The method proposed permits a generalization to the case of any rational root and can be also used for the calculation of the DWSG coefficients for the operator $\sqrt{-\nabla^2} + V(x)$ whose the square is not the Laplace operator.

3   A generalization to the case of an arbitrary rational root and the operator which cannot be presented as a power of the Laplace operator

In this section we generalize the method proposed to the case of an arbitrary rational root, i.e. for the operator $(-\nabla^2 + V(x))^{p/m}$, where p and m are any

naturals. For the sake of simplicity, we actually consider the case of a natural root, i.e., the operator of the type $(-\nabla^2 + V(x))^{1/m}$, where m is any natural number and show that the method can be easy generalized to the case of an arbitrary rational root. The equation for σ has the form

$$\left((-(\nabla_\mu + i\nabla_\mu l)(\nabla^\mu + i\nabla^\mu l) + V(x))^{1/m} - \lambda\right)\sigma(x, x', k; \lambda) = \mathrm{I}(x, x'). \quad (35)$$

To generate the heat kernel expansion, we introduce an auxiliary parameter ϵ into eq. (35) according to the rule $l \to l/\epsilon^{m/2}$, $\lambda \to \lambda/\epsilon$ and expand the amplitude in a formal series in the powers of ϵ

$$\sigma(x, x', k; \lambda) = \sum_{s=0}^{\infty} \epsilon^{1+s\frac{m}{2}} \sigma_s(x, x', k; \lambda). \quad (36)$$

We seek an expansion of the root in the form

$$(\nabla_\mu l \nabla^\mu l - i\epsilon^{m/2}\nabla^2 l - \epsilon^m \nabla^2 - 2i\epsilon^{m/2}\nabla_\mu l \nabla^\mu +$$

$$\epsilon^m V(x))^{1/m} = (\nabla_\mu l \nabla^\mu l)^{1/m} + \epsilon^{m/2} f_1 + \epsilon^m f_2 + \dots \quad (37)$$

Taking the mth power of eq. (37), we obtain the following equations for the unknown f_1 and f_2:

$$-i\nabla^2 l - 2i\nabla_\mu l \nabla^\mu = \sum_{i=0}^{m-1} R^i f_1 R^{m-1-i}, \quad (38)$$

$$-\nabla^2 + V(x) = \sum_{i=0}^{m-2}\sum_{j=i}^{m-2} R^i f_1 R^{j/2} f_1 R^{m-2-i-j} + \sum_{i=0}^{m-1} R^i f_2 R^{m-1-i}, \quad (39)$$

where $R = (\nabla_\mu l \nabla^\mu l)^{1/m}$. Note that in the case of an arbitrary ratioanl root, i.e., for the operators of the type $(-\nabla^2 + V(x))^{p/m}$, we would be have the pth power of $-i\nabla^2 l - 2i\nabla_\mu l \nabla^\mu$ on the left-hand side of equation (37) that does not present an untractable problem for generating the expansion in the powers of ϵ. All the same we again can write down a general structure in derivatives of the root and taking the mth power can find the unknowns in the expansion of the root. Thus, the method can be used in the case of an arbitrary rational root. Writting down general structures of f_1 and f_2

$$f_1 = -i(a_\mu \nabla^\mu + b),$$

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$$f_2 = C_1 \nabla^2 + C_{2\mu} \nabla^\mu + C_{3\mu\nu} \nabla^{mu\nu} + C_4, \tag{40}$$

from eqn. (38) and (39) we find the explicit expressions for f_1 and f_2 and from eq. (35) obtain the recursion relations for σ_0, σ_1 and σ_2

$$(R - \lambda)\sigma_0 = I,$$

$$(R - \lambda)\sigma_1 - i(a_\mu \nabla^\mu + b)\sigma_0 = 0,$$

$$(R - \lambda)\sigma_2 - i(a_\mu \nabla^\mu + b)\sigma_1 + (C_1 \nabla^2 + C_{2\mu} \nabla^\mu +$$

$$C_{3\mu\nu} \nabla^\mu \nabla^\nu + C_4)\sigma_0 = 0, \tag{41}$$

where $R = (\nabla_\mu l \nabla^\mu l)^{1/m}$. Taking the coincidence limits, we have

$$[\sigma_0] = \frac{1}{(k^2)^{1/m} - \lambda},$$

$$[\sigma_2] = -\frac{2k_\mu k^\lambda l^{\mu\nu}{}_{\nu\lambda}}{m^2 (k^2)^{2-2/m}((k^2)^{1/m} - \lambda)^3} - -\frac{2k_\mu k_\lambda l_{nu}^{\nu\mu\lambda}}{m^2 (k^2)^{2-2/m}((k^2)^{1/m} - \lambda)^3} -$$

$$\frac{V(x)}{m(k^2)^{1-1/m}((k^2)^{1/m} - \lambda)^2} -$$

$$\frac{(m-1)k_\mu k^\lambda l^{\mu\nu}{}_{\nu\lambda}}{m^2 (k^2)^{2-1/m}((k^2)^{1/m} - \lambda)^3} - \frac{(m-1)k^\mu k^\lambda l_\nu^{\nu\mu\lambda}}{m^2 (k^2)^{2-1/m}((k^2)^{1/m} - \lambda)^3} \tag{42}$$

where we write down only terms which do not vanish after the substitution of the explicit expression for $l_{\mu\ldots\lambda}$ and the convolution with $k^{\mu\ldots\lambda}$.

Integrating in λ and k, we obtain the lowest DWSG coefficients of the heat kernel expansion for the operator $(-\nabla^2 + V(x))^{1/m}$

$$E_0(x) = \frac{m\Gamma(\frac{m}{2}n)}{(4\pi)^{n/2}\Gamma(n/2)}, \tag{43}$$

$$E_2(x) = \frac{m\Gamma\left(m(\frac{n-m}{2})\right)}{(4\pi)^{n/2}\Gamma(\frac{n-m}{2})}\left(\frac{R}{6} - V(x)\right). \tag{44}$$

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E_0 and E_2 calculated in the case of an arbitrary natural root m coincide with E_0, E_2 calculated in the particular case m=2 (see eqn. (32) and (34)). Note that in comparision with the case of nonmiminimal operators [9] the dependence on m and the dimension of space is rather trivial, namely, E_2 depends on m only through a constant factor.

We now show how the method proposed works in the case of the calculation of the DWSG coefficients for the operator $\sqrt{-\nabla^2} + V(x)$ which cannot be obviously represented as a power of the Laplace operator. We can use the old expansion (25) for the root $\sqrt{\nabla_\mu l \nabla^\mu l - i\epsilon \nabla^2 l - \epsilon^2 \nabla^2 - 2i\epsilon \nabla_\mu l \nabla^\mu}$. Further, as usually, in order to generate the heat kernel expansion, we introduce an auxiliary parameter ϵ according to the rule $l \to l/\epsilon$, $\lambda \to \lambda/\epsilon$ and expand the amplitude σ in a formal series in powers of ϵ

$$\sigma_\epsilon(x, x', k; \lambda) = \sum_{m=0}^{\infty} \epsilon^{1+m} \sigma_m(x, x', k; \lambda). \tag{45}$$

Then, the equations for $\sigma_0, \sigma_1, \sigma_2$ take the form

$$(R^{1/2} - \lambda)\sigma_0 = I,$$

$$(R^{1/2} - \lambda)\sigma_1 + (-i(a_\mu \nabla^\mu + b) + V(x))\sigma_0 = 0,$$

$$(R^{1/2} - \lambda)\sigma_2 + (-i(a_\mu \nabla^\mu + b) + V(x))\sigma_1$$
$$+(C_1 \nabla^2 + C_{2\mu} \nabla^\mu + C_{3\mu\nu} \nabla^\mu \nabla^\nu + C_4)\sigma_0 = 0. \tag{46}$$

From eqn.(46) we find

$$[\sigma_1] = -\frac{V(x)}{(\sqrt{k^2} - \lambda)^2},$$

$$[\sigma_2]_{new} = \frac{k_\mu \nabla^\mu V(x)}{(\sqrt{k^2} - \lambda)^3 \sqrt{k^2}} + \frac{V^2(x)}{(\sqrt{k^2} - \lambda)^3}, \tag{47}$$

where we write down only the terms with $V(x)$; the terms with $l_{\mu...\alpha}$ coincide with that for the operator $\sqrt{-\nabla^2} + V(x)$. The first term in the expression for $[\sigma_2]_{new}$ vanishes after the integration in k because of an odd power of

k. Note also that in difference to the case of the operator $\sqrt{-\nabla^2 + V(x)}$ coefficient E_1 is not equal to zero for the operator $\sqrt{-\nabla^2} + V(x)$.

Integrating in λ and k, we obtain

$$E_1(x) = -\frac{2\Gamma(n)V(x)}{(4\pi)^{n/2}\Gamma(n/2)}, \tag{48}$$

$$E_2(x)_{new} = -\frac{2\Gamma(n)V^2(x)}{(4\pi)^{n/2}\Gamma(n/2)}. \tag{49}$$

Consequently, the entire E_2 coefficient is

$$E_2(x) = \frac{\Gamma(n-1)}{(4\pi)^{n/2}\Gamma(n/2)}(\frac{R}{6} - 2(n-1)V^2(x)). \tag{50}$$

Note that contrary to the case of the operator $\sqrt{-\nabla^2 + V(x)}$ the E_2 coefficient for the operator $\sqrt{-\nabla^2} + V(x)$ essentially depends on the dimension of space. We can also generalize the method proposed to the case of the operator of the type $(-\nabla^2)^{1/m} + V(x)$, where m is any natural number. Thus, we have shown that the method proposed can be modified and adopted for the calculation of the DWSG coefficients for various operators which involve the extraction of root and have calculated lowest E_2 coefficient for three different operators. Amount of work needed to calculate the DWSG coefficients icreases very quickly with the growth of the order of the DWSG coeffcient in the method proposed. It is connected with rapid increase of the number of terms in the expansion of the root with the growth of the order of the DWSG coefficient. In fact, in the case where the operator under the sign of the root is strictly positive there exists a more simple and less laborious method for the calculation of the DWSG coefficients.

4   More simple method for the calculation of the DWSG coefficients for strictly positive operators

Let us again consider the operator $A = \sqrt{-\nabla^2 + V(x)}$. If this operator is strictly positive, i.e., it does not have any zero eigenmodes, we can used

instead of (6) the following representation:

$$e^{-tA} = \int_C \frac{id\lambda}{2\pi} e^{-t\lambda^{1/2}} (A - \lambda)^{-1}, \tag{51}$$

We demand that $-\nabla^2 + V(x)$ do not have any zero eigenmodes because in such a case the contour C can be drawn such that it encircles the whole spectrum of the operator $-\nabla^2 + V(x)$ and do not intersect anywhere the cut from infinity to zero along the negative half-axis which is needed in order that the extraction of root be meaningful. If the operator has zero eigenmodes, such a countour cannot be drawn because in this case we cannot draw the countour in such a way that it do not intersect the cut and simultaneously the contribution of eigenmodes be properly taken into account. By using this method, we can prove that the DWSG coefficients for operators with root are expressed through the DWSG coefficients for operators without root. Note that this method cannot be used for operators which have zero eigenmodes. However, the method with the expansion of root can be use also in the case of operators with zeromodes. Of course, for strictly positive operators both methods can be used and they yield coinciding results.

E_m coefficients in the method with the representation $e^{-t\lambda^{1/2}}$ are given by the relation

$$E_m(x|A) = \int \frac{d^n k}{(2\pi)^n \sqrt{g}} \int_C \frac{id\lambda}{2\pi} e^{-\lambda^{1/2}} [\sigma_m](x, x, k; \lambda), \tag{52}$$

where $\sigma_m(x, x, k; \lambda)$ are the same as for the operator $-\nabla^2 + V(x)$. Using σ_2 obtained in work [6] and calculating the integrals over λ and k, we obtain the following E_2 coefficient for the operator $\sqrt{-\nabla^2 + V(x)}$:

$$E_2(x) = \frac{2\Gamma(n-2)}{(4\pi)^{n/2-1}\Gamma(n/2)} \left(\frac{R}{6} - V(x) \right) \tag{53}$$

which coincides with E_2 obtained in Section 2 by using the method with the expansion of root.

Let us show by using the method with representation (51) that the DWSG coefficients for operators with root are explicitly expressed through the DWSG coefficients for operators without root. Let us consider the DWSG

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coefficients for the operator $-\nabla^2 + V(x)$. According to [6], they are given by the relation

$$E_m(x|A) = \int \frac{d^n k}{(2\pi)^n \sqrt{g}} \int_C \frac{id\lambda}{2\pi} e^{-\lambda} [\sigma_m](x, x, k; \lambda). \tag{54}$$

Comparing it with (52), we see that the only difference is the power of λ in the exponent, namely, it is equal to $1/2$ in the case of the operator with root and 1 for the operator without root. We recall that a general term of $[\sigma(x, x, k; \lambda)]$ has the following form:

$$\frac{k_{\mu_1} \ldots k_{\mu_{2s}} F^{\mu_1 \ldots \mu_{2s}}}{(k^2 - \lambda)^a}, \tag{55}$$

where $F^{\mu_1 \ldots \mu_{2s}}$ is expressed through the bundle curvature $W_{\mu\nu}$ and the Riemannian curvature tensor $R^\lambda_{\rho\mu\nu}$. $[\sigma_m(x, x, k; \lambda)]$ is the sum of terms with various powers of a and s. It is very important for what follows that the difference $a - s$ is fixed for the DWSG coefficient of a given order. This fact follows from the homogeneity property of the reccurent relations for σ_m (see [6]). $a - s$ is equal to $1 + m/2$ for the operator $-\nabla^2 + V(x)$. Integrating over λ and angles in n-dimensional space, we have for the DWSG coefficient in the case of the operator without root

$$\int dk k^{n-1+2s} g_{\{\mu_1 \ldots \mu_{2s}\}} F^{\mu_1 \ldots \mu_{2s}} \frac{d^{a-1}}{dk^{2(a-1)}} e^{-k^2} \tag{56}$$

and for the operator with root

$$\int dk k^{n-1+2s} g_{\{\mu_1 \ldots \mu_{2s}\}} F^{\mu_1 \ldots \mu_{2s}} \frac{d^{a-1}}{dk^{2(a-1)}} e^{-k}, \tag{57}$$

where we have omitted common constant factors which coincide for two cases under consideration and have used formula (31). Integrating over k, we obtain $\Gamma(\frac{n-2}{2} + s - a + 2) = \Gamma(\frac{n-m}{2})$ for the operator without root and $2\Gamma(n - 2 + 2s - 2a + 4) = 2\Gamma(n - m)$ for the operator with root. It is very essential that the Γ-functions do not depend on a and s due to the homogeneity property and depend only on m. Therefore, the results obtained are true for any term in the expansion of σ_m. Thus, the DWSG coeffients for the operator with root are expressed through the DWSG coeffients for the operator without root

$$E_{mr} = \frac{2\Gamma(n - m)}{\Gamma(\frac{n-m}{2})} E_m, \tag{58}$$

where E_{mr} are the DWSG coefficients for the operator with root. It is easy to check that E_0 and E_2 directly calculated in Section 2 for the operator $\sqrt{-\nabla^2 + V(x)}$ by using the method with the expansion of root (see formulas (31) and (34)) coincide with the DWSG coefficients given by the common formula (58). By using the representation with $e^{\lambda^{p/q}}$, similarly, it is easy to show that the DWSG coeffcents for the operator with an arbitrary rational root, i.e., for operators of the type $(-\nabla^2 + V(x))^{p/q}$, where p and q are any natural numbers, are expressed through the DWSG coeffcents for the operator without root as follows:

$$E_{mr} = \frac{q/p\Gamma(q/p\frac{n-m}{2})}{\Gamma(\frac{n-m}{2})}E_m. \tag{59}$$

In the particular case of the square root, this formula yields (54) and in the case of natural root, i.e., $p = 1$, E_0 and E_2 given by (59) coincide with E_0 and E_2 explicitly calculated in Section 3, formulas (43) and (44). Thus, the problem of finding of the DWSG coefficients for operators of the type of rational root of a strictly positive operator is exactly solvable, i.e., the DWSG coefficients for operators with root are explicitly expressed through those for operators without root.

Note that it would be of significant interest to calculate the DWSG coefficients by using two methods proposed for an operator which has eigenmodes. Finding the difference between E_m obtained by two methods, we would be able to define the contribution of eigenmodes to the DWSG coefficients.

5   Acknowledgments

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